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A NONSTANDARD CHARACTERIZATION OF SUBINVARIANT MEASURES.(U)
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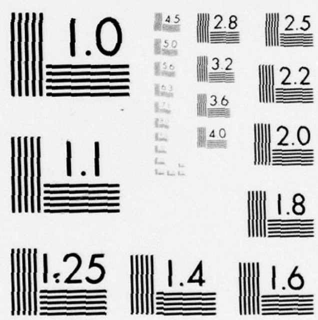

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A NONSTANDARD CHARACTERIZATION
OF SUBINVARIANT MEASURES

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Introduction

The problem of constructing a subinvariant measure on a collection of semi-group homeomorphisms from the unit interval to itself arises in Mathematical Economics in the area of utility orderings over time (Koopmans [1], Koopmans, Diamond, and Williamson [2], and Koopmans and Williamson [3].) The existence of such a measure is indicative of what Koopmans et al. term weak time perspective, and signifies a certain kind of impatience with respect to the time available date of desirable consumption commodities.

As an illustration of the method of proof by nonstandard analysis as applied to Mathematical Economics, we provide an alternative characterization of such measures below. The construction is based on the work of Hausner [5] and Parikh [6].

To motivate an appreciation of the construction, we provide an exposition of the framework developed by Koopmans [1] and subsequently modified and elaborated upon by Koopmans et al. [2] and Koopmans [4].

I. The Koopmans Framework

Consider a space, X, consisting of infinite sequences, $(x_{(1)}, x_{(2)}, \ldots, x_{(n)}, \ldots)$, the elements of which, $x_{(t)}$, are each members of a finitely dimensioned Euclidean space, R^n_+. The members of X are termed programmes, and X is termed the programme space. Notationally, ${}_tx$, for $t \in N$, will mean the entries of a programme from time t onwards. That is, ${}_tx = (x_{(t)}, x_{(t+1)}, \ldots)$ and x_t will mean the truncation of a programme to t time periods, that is, $x_t = (x_{(1)}, \ldots, x_{(t)})$.

The following set of postulates is given in Koopmans, Diamond and Williamson [2] to characterize that the authors term a quasi-cardinal utility representation on X. A weakened version of these postulates can be found in Koopmans [4].

Postulate I: On the assumption that the elements of X are each from a compact subset of R^n_+, Ω, there exists a utility function $U({}_1x)$ defined for all ${}_1x \in X$ such that $U({}_1x)$ is continuous in the following sense: If $\tilde{U}$ is a value of $U({}_1x)$ on X, and if $\tilde{U}'$ and $\tilde{U}''$ are such that $\tilde{U}' < \tilde{U} < \tilde{U}''$, then there exists a positive real, δ, such that $\tilde{U}' \leq U({}_1x') \leq \tilde{U}''$ for all ${}_1x' \in X$ such that $d({}_1x', {}_1x) \leq \delta$. The utility function $U(\cdot)$ is ordinal in character.

Postulage II: There exist x_1 and x_1' and ${}_2x$ such that $U(x_1, {}_2x) > U(x_1', {}_2x)$.

Postulate III: For any x_1, x_1', ${}_2x$, ${}_2x'$,

$$U(x_1, {}_2x) \geqq U(x_1', {}_2x) \Rightarrow U(x_1, {}_2x') \geqq U(x_1', {}_2x')$$

$$U(x_1, {}_2x) \geqq U(x_1, {}_2x') \Rightarrow U(x_1', {}_2x) \geqq U(x_1', {}_2x')$$

Postulate IV: For some x_1 and all ${}_2x$, ${}_2x'$,

$$U(x_1, {}_2x) \geqq U(x_1, {}_2x') \text{ if and only if } U({}_2x) \geqq U({}_2x')$$

Postulate V: For some ${}_1\underline{x}$ and ${}_1\overline{x}$,

$$U({}_1\underline{x}) \leqq U({}_1x) \leqq U({}_1\overline{x}) \text{ for any } {}_1x \in X.$$

The metric used to induce the topology in which $U(\cdot)$ is presumed to be continuous by Postulate I is the sup norm:

$$d({}_1x', {}_1x) = \sup_t |x_t' - x_t| \text{ for}$$

$$|x_t' - x_t| = \max_j |x_{tj}' - x_{tj}| \text{ for } j = 1, \ldots, n$$

Postulates I through IV have been shown to imply that there exist auxiliary functions $u(\cdot)$ and $V(u,U)$ such that for ${}_1x \in X$

(i) $U({}_1x) = V\big(u(x_1), U({}_2x)\big)$

and

(ii) $V_T({}_1u_T; U) = V\big(u_1, V\, u_1, V(u_2, \ldots, V(u_T, U)\big)$

where ${}_1u_T = (u_1, u_2, \ldots, u_T)$ and $u_j = \big(x(j)\big)$ for $j = 1, \ldots, T$, where each u_j is continuous in the sense of Postulate I, and V_T is the T^{th} iteration of the operator V.

The demonstration of the above relations can be found in Koopmans [1], pp. 292-295.

For a truncation of ${}_1x$ to x_T, the u_j for $j = 1, \ldots, T$ represent the immediate utility levels of the successive

entries of ${}_1x$, $x_{(1)}, \ldots, x_{(T)}$, while the value of $U({}_tx)$ is the prospective utility evaluation of ${}_1x$ from time period t onwards. The aggregator function, $V(u,U)$, serves to indicate that temporal evaluation of members of X can be decomposed into immediate and remote utilities with respect to a fixed time period. As a generalzation of (i), one has

(iii) $U({}_1x) = V_T\left({}_1u_T, U({}_{T+1}x)\right)$

with the further interpretation that the postponement of a programme of utility U by T time periods is precisely compensated for by the interpolation, in the T periods in between, of commodity vectors $x_{(1)}, \ldots, x_{(T)}$ with utility levels $u_1(x_{(1)}), \ldots, u_T(x_{(T)})$.

By Postulate V, the range of the functions $U(\cdot)$ and $u(\cdot)$ can be assumed to be the closed interval, $[0,1]$, so that one has:

(iv) $0 = U({}_1\underline{x}) \leq U({}_1x) \leq U({}_1\overline{x}) = 1$ for programmes ${}_1x$

(v) $0 = u(\underline{x}_{(j)}) \leq u(x_{(j)}) \leq u(\overline{x}_{(j)}) = 1$ for vectors $x(j)$

Then one has as a consequence, for the auxiliary function $V(u,U)$:

(vi) $V\left(u(\underline{x}_1), U({}_2\underline{x})\right) = 0$ and $V\left(u(\overline{x}_1), U({}_2\overline{x})\right) = 1$

The range of $V(u,U)$ is then the unit square, $[0,1]^2$.

It is further demonstrated that for a fixed specification of vectors, dependent on T, ${}_1\underline{x}_T = (x_{(1)}, \ldots, x_{(T)})$, such that, given the associated utility pattern, ${}_1u_T = \left(u_1(x_{(1)}, \ldots, u_T(x_{(T)})\right)$ of immediate evaluations per

each time period, there is a unique value

(vii) $U = W_T({}_1u_T)$

such that (iii) is satisfied, where $W_T({}_1u_T)$ is a continuous function of the u_j, $j = 1, \ldots, T$ ([1], pp. 297-300).

The concept of temporal perspective is defined in terms of the auxiliary function, $V(u,U)$, as applied to the range of the utility scale, which by Postulate V is the unit interval, [0,1] ([2], p. 88). The following is a statement of this property.

(γP) For two values, $U', U \varepsilon [0,1]$, if $U' > U$ and $V_T({}_1u_T, U) = U''$ and $V_T({}_1u_T, U') = U'''$ for $T \geq 1$, then weak and strong temporal perspectives are

(WγP) Df: $U''' - U'' \leq U' - U$

(SγP) Df: $U''' - U'' < U' - U$

As the comparisons are those of utility differences, Koopmans, Diamond and Williamson have used the term quasi-cardinal when referring to the above property ([2], p. 88).

Alternatively, one might consider points U' on the scale of utility as being imbedded in closed non-empty intervals. For example, a typical interval is of the form:

$\mathbb{U} = [\underline{U}, \overline{U}] = \{U : \underline{U} \leq U \leq \overline{U}\}$

Following convention, we denote the unit interval as:

$I = [0,1]$

In the context of interval representations, the following can also be defined in terms of set theoretic inclusion:

(viii) $\mathbb{U}' \supset \mathbb{U}''$ Df: $\underline{U}' \leq \underline{U}'' < \overline{U}'' \leq \overline{U}'$

From the continuity of the auxiliary functions $V(u,U)$ in U, and from the monotonicity implied by the relation in (iii), one has the following, when $V(u,\cdot)$ is applied to an interval:

(ix) $V(u,\mathbb{U}) = [V(u,\underline{U}),V(u,\overline{U})]$

Then by the recursive character of $V(u,U)$ given in (iii) also, one can apply $V(u,U)$ in iteration for a finite sequence of values ${}_1u_T = (u_1, \ldots, u_T)$ on programmes whose utility values lie in a fixed interval to obtain the following:

(x) $\mathbb{U}' = V\mathbb{U}$ Df: $V_T({}_1u_T,\mathbb{U}) = \mathbb{U}'$ for a time period $T \geq 1$ and a sequence of immediate utility evaluations ${}_1u_T = (u_1, \ldots, u_T)$

It is then permissible to interpret V as an operator acting on the closed subsets of I, from the implications of Postulate V. Let [V] denote the class of auxiliary functions $V(\cdot,\cdot)$. It is desirable that members of [V] satisfy the following properties:

(a) [V] is a semi-group. For $V_1,V_2 \in [V]$ then $(V_1 \circ V_2) \in [V]$.

(b) Members of [V] transform I onto subintervals, and are pointwise continuous on X. For $V \in [V]$, $VI \subset I$.

(c) If U' and U'' are interior values of I, then for some $V \in [V]$, $VU' = U''$.

(d) [V] is a subinvariant class with respect to I. No member of [V] takes an interval $\mathbb{U}$ into an interval $\mathbb{U}'$ that contains $\mathbb{U}$. For any $V \varepsilon [V]$ if $V\mathbb{U} = \mathbb{U}'$, then $\mathbb{U}' \not\supset \mathbb{U}$.

The last condition, (d), is necessary for temporal perspective to be exhibited. If (d) were false, and for some $V \varepsilon [V]$ and some interval $\mathbb{U}, \mathbb{U} \subseteq V\mathbb{U}$, then there can be no scale in which $V\mathbb{U}$ is less than $\mathbb{U}$, and $(S\gamma P)$ of (γP) would not hold.

Condition (a) can be derived from the expression in (x) and the fact that the future is temporally asymmetric – it has a beginning but no end. Then for some $V_1, V_2 \varepsilon [V]$, if $\mathbb{U}' = V_1\mathbb{U}$ and $\mathbb{U}'' = V_2\mathbb{U}'$ for some interval $\mathbb{U}$, then $\mathbb{U}'' = V_2(V_1\mathbb{U})$. From the definition given in (x), it follows that for some $T_1, T_2 \geq 1$ and ${}_1u^1_{T_1}, {}_1u^2_{T_2}$,

$$\text{(xi)} \quad \mathbb{U}'' = V_{T_2}\left({}_1u^2_{T_2}, V_{T_2}({}_1u^1_{T_1}, \mathbb{U})\right) = V_{T_1+T_2}({}_1u^1_{T_1}, {}_1u^2_{T_2}, \mathbb{U})$$

II. The Nonstandard Construction

In this section, we will require that M^* be an $\aleph_1$-saturated enlargement of a set theoretical structure M, sufficiently rich for the real number system, as found in Reference 8 or 9. The specific features of M^* we will make use of in the construction that follows are those of concurrence and transfer, which for the sake of convenience we state as lemmas.

Definition II.1: A relation ρ is said to be concurrent in M if $\rho \in M$ and if whenever $x_1, \ldots, x_n$ are elements of $\ell D(\rho)$ there is an element b in $\mathfrak{R}(\rho)$ such that $\langle x_i, b \rangle \in \rho$ for $i = 1, \ldots, n$, where n is a standard natural number, i.e., $n \in N$.

▶Lemma II.2: (Concurrence Theorem) Let ρ be a concurrent relation in M. Then there exists an element $b \in M^*$ such that $\langle x^*, b \rangle \in \rho^*$ for all $x \in \ell D(\rho)$.

Proof: Davis [8], Theorem 8.1 of Ch. I.

▶Lemma II.3: (Principle of Transfer) Denote by L_M and L_{M^*} the first-order languages of the standard and nonstandard universes corresponding to M and M^*, respectively, and let Σ be a sentence of L_M.

Then $^*\!\models \Sigma^*$ if and only if $\models \Sigma$.

Proof: Davis [8], Theorem 7.3 of Ch. I, and Theorems 8.2, 8.3 and 8.5 of Ch. I as well.

We shall consider the unit interval endowed with its natural topology, the open sets being open subintervals. Notationally, O_x will denote an open neighborhood of some $x \in I$, and $\{O_x\}$ will denote the collection of open neighborhoods of x. In this context, the collection, [V], is to be considered as a semi-group of homeomorphisms on $\mathcal{B}(I)$, the Borel sets of I. We will make use of the following non-standard topological concepts.

Definition II.4: For a given point $x \in I$, define the monad of x, symbolized $\hat{u}(x)$, to be the following:

$$\hat{u}(x) = \cap\{\mathcal{U}^* : \mathcal{U} \in \{O_x\}\}$$

If for some $q \in I^*$, if $q \in \hat{u}(x)$, then we symbolize that $(x = q)_{\text{Mod } M_1}$, where $M_1 = \{r \in R^* : |r| < 1/\omega,\ \omega \in N^* - N\}$. M_1 is the set of infinitesimals in R^*.

Lemma II.5: Let $x \in I$. Then there is an internal set $G \in \{O_x^*\}$ such that $G \subseteq \hat{u}(x)$.

Proof: (Robinson [9], Theorem 4.1.2.)

Let $\rho = \{\langle z,y\rangle : z \in \{O_x\}, y \in \{O_x\} \wedge y \subseteq z\}$. We show that ρ is concurrent. It is obvious that $\ell D(\rho) = \{O_x\}$. Let $\mathcal{U}_1, \ldots, \mathcal{U}_n \in \{O_x\}$; then in the natural topology,

$\left(\bigcap_{j=1}^{n} \mathcal{U}_j\right) \varepsilon \{O_x\}$, and thus $<\mathcal{U}_j, Q> \varepsilon\, \rho$ for all $j = 1, \ldots, n$ for $n \varepsilon N$ and $Q = \left(\bigcap_{j=1}^{n} \mathcal{U}_j\right)$, indicating the concurrency of ρ.

By Lemma II.2, there is a $G \varepsilon M^*$ such that for all $\mathcal{U} \varepsilon \{O_x\}$, $<\mathcal{U}^*, G> \varepsilon\, \rho^*$. Since it is true that $\rho^* \subseteq \{O_x^*\} \times \{O_x^*\}$, by Lemma II.3,

$$* \models (\forall z^* \varepsilon \{O_x^*\})(\forall y^* \varepsilon \{O_x^*\})(<z^*, y^*> \varepsilon\, \rho^* \Rightarrow y^* \subseteq z^*)$$

and we are permitted to infer that $G \subseteq \mathcal{U}^*$ and consequently that $G \subseteq \cap \{\mathcal{U}^* : \mathcal{U} \varepsilon \{O_x\}\} = \hat{u}(x)$.

Q.E.D.

Lemma II.6: A set $Q \subseteq I$ is open if and only if for all $x \varepsilon Q$, $\hat{u}(x) \subseteq Q^*$.

Proof: (Robinson [9], Theorem 4.1.4)

Let Q be open and $x \varepsilon Q$. Then since $Q \varepsilon \{O_x\}$ and $\hat{u}(x) = \cap \{\mathcal{U}^* : \mathcal{U} \varepsilon \{O_x\}\}$, $\hat{u}(x) \subseteq Q^*$.

On the other hand, let $x \varepsilon Q$ such that $\hat{u}(x) \subseteq Q^*$. By Lemma II.5, there is some internal $G \varepsilon \{O_x^*\}$ such that $G \subseteq u(x) \subseteq Q^*$. Then by Lemma II.3, one can obtain by virtue of $* \models (\exists G \varepsilon \{O_x^*\})(G \subseteq Q^*)$, that for some $\mathcal{U} \varepsilon \{O_x\}$, $\mathcal{U} \subseteq Q$. Then if $\hat{u}(x) \subseteq Q^*$, for $x \varepsilon I$, for some $\mathcal{U} \subseteq I$, $\mathcal{U} \varepsilon \{O_x\}$, which we fix as $\mathcal{U}_x$. Then $Q = \bigcup_{x \varepsilon Q} \mathcal{U}_x$.

Q.E.D.

Lemma II.7: Let the function f, map I into R. Then f is continuous at $\tilde{x} \in I$ if and only if for all $x^* \in I^*$, $(x^* = \tilde{x}^*)_{\text{Mod } M_1}$ implies that $(f^*(x^*) = f^*(\tilde{x}^*))_{\text{Mod } M_1}$.

Proof: (Robinson [9], Theorem 3.4.5)

If f is continuous at $\tilde{x} \in I$, then for $e > 0$, $\delta > 0$ in R, one has

$$\Vdash (\forall x \in I)\left(|x - \tilde{x}| < \delta \Rightarrow |f(x) - f(\tilde{x})| < e\right)$$

Then by Lemma II.3, if $x^* \in I^*$ and $(x^* = \tilde{x}^*)_{\text{Mod } M_1}$, certainly $|x^* - \tilde{x}^*| < \delta \in R$ and for arbitrary $e \in R$ $|f^*(x^*) - f^*(\tilde{x}^*)| < e$, then $(f^*(x^*) = f^*(\tilde{x}^*))_{\text{Mod } M_1}$ by the arbitrariness of e.

Now, if $(x^* = \tilde{x}^*)_{\text{Mod } M_1}$ implies $(f^*(x^*) = f^*(\tilde{x}^*))_{\text{Mod } M_1}$, then choose $\delta \in M_1^+ - \{0\}$ for a fixed $e \in R$, $e > 0$, and standard, then

$$*\Vdash (\exists \delta \in R_+^*)(\forall x^* \in I^*)\left(|x^* - \tilde{x}^*| < \delta \Rightarrow |f^*(x^*) - f^*(\tilde{x}^*)| < e\right)$$

By Lemma II.3, once more, f is continuous at $\tilde{x}$.

Q.E.D.

We turn now to the demonstration of the principal result, and remark that the construction utilizes methods employed by Hausner [5] and Parikh [6] to construct Haar measures on the real line.

Definition II.8: Let $\tilde{x}$ be interior to I, then $\mathbb{U}_{\tilde{x}} \in \{O_{\tilde{x}}\}$ is said to separate $\mathbb{A}$ from B under [V], for $A, B \in \mathbb{B}(I)$ if for any $V \in [V]$ it is the case that either $\mathbb{A} \cap V\mathbb{U}_{\tilde{x}} = \phi$ or $B \cap V\mathbb{U}_{\tilde{x}} = \phi$.

$\mathfrak{A}$ and B are separated in I under [V] if for some $\tilde{x} \varepsilon I$ there exists some $\mathfrak{U}_{\tilde{x}}$ that separates them.

Definition II.9: Let ψ be a set valued mapping on $\mathfrak{B}(I)$, taking values in R, then ψ is said to be admissible if it satisfies the following:

(1) $\psi(\mathfrak{A}) \geq 0$ for any $A \varepsilon \mathfrak{B}(I)$

(2) $\mathfrak{A} \subseteq B$ implies $\psi(A) \leq \psi(B)$ for any $\mathfrak{A}, B \varepsilon \mathfrak{B}(I)$

(3) $\psi(\mathfrak{A} \cup B) \leq \psi(\mathfrak{A}) + \psi(B)$ for any $\mathfrak{A}, B \varepsilon \mathfrak{B}(I)$

Definition II.10: Let ψ be as in Definition II.9, then ψ is weakly additive if it satisfies

(4) $\psi(\mathfrak{A}) + \psi(\mathfrak{B}) = \psi(\mathfrak{A} \cup \mathfrak{B})$ for any $\mathfrak{A}, B \varepsilon \mathfrak{B}(I)$

whenever $\mathfrak{A}$ and $\mathfrak{B}$ are separated.

We give now a statement of the main theorem, the proof of which shall proceed by a series of lemmata.

▶Theorem II.11: Let [V] be a semi-group of homeomorphisms on the Borel sets of $I = [0,1]$, $\mathfrak{B}(I)$, such that

(a) For any $V \varepsilon [V]$ and $\mathfrak{U} \varepsilon \mathfrak{B}(I)$, $V\mathfrak{U} \not\supseteq \mathfrak{U}$.

(b) For any $\mathfrak{U} \varepsilon \mathfrak{B}(I)$ and $\tilde{x} \varepsilon I$, there is some $V \varepsilon [V]$ such that $\tilde{x} \varepsilon V\mathfrak{U}$.

Then there exists a nonnegative, weakly additive, real valued set function, λ, on the F_σ sets of $\mathfrak{B}(I)$ such that $\lambda(\mathfrak{U}) \geq \lambda(V\mathfrak{U})$ for $V \varepsilon [V]$ and any $\mathfrak{U} \varepsilon \mathfrak{B}(I)$.

Proof: We proceed by a series of lemmata.

Lemma II.11.1: For a fixed $\tilde{x} \in I$, let $\mathfrak{U} \in \{O_{\tilde{x}}\}$, then for any $V \in [V]$, $V\mathfrak{U}$ is an open set.

Proof: Let $V\tilde{U} \in V\mathfrak{U}$ for $\tilde{U} \in \mathfrak{U}$. Since $\mathfrak{U} \in \{O_{\tilde{x}}\}$, by Lemma II.6, $\hat{u}(\tilde{U}) \subseteq \mathfrak{U}^*$. To show that $V\mathfrak{U}$ is open, one needs to show that $\hat{u}(V\tilde{U}) \subseteq V^*\mathfrak{U}^*$. However, since each $V \in [V]$ is a homeomorphism on I, this follows from Lemma II.7.

Q.E.D.

Consider next some $\mathfrak{A} \in \mathcal{B}(I)$ and for a fixed $\tilde{x}$, some $\mathfrak{U}_{\tilde{x}} \in \{O_{\tilde{x}}\}$. Since I is locally compact, $\mathfrak{A} \subseteq F$ for some $F \in F_\sigma$. By the above lemma, $V\mathfrak{U}_{\tilde{x}}$ is an open set for each $V \in [V]$. Then by Condition (b) of the premises, $\bigcup_{V \in [V]} V\mathfrak{U}_{\tilde{x}} = I$; and thus $\{V\mathfrak{U}_{\tilde{x}} : V \in [V]\}$ is an open covering of $F \subseteq I$. Since F is an F_σ, F is compact on I and therefore for some finite collection $\{V_1, \ldots, V_n\}$, where n is a standard natural number; since $\mathfrak{A} \subseteq F$, $\mathfrak{A} \subseteq \bigcup_{j=1}^{n} V_j\mathfrak{U}_{\tilde{x}}$, and denote by $(\mathfrak{A} : \mathfrak{U}_{\tilde{x}})$ the least value in N for which this is true. Let $\mathfrak{A}_0$ be fixed and non-degenerate in $\mathcal{B}(I)$. Define the set valued function Φ on $\mathcal{B}(I)$ as follows:

$$\Phi(\mathfrak{A}, \mathfrak{U}_x) = \Phi_{\mathfrak{U}_{\tilde{x}}}(\mathfrak{A}) = \frac{(\mathfrak{A} : \mathfrak{U}_{\tilde{x}})}{(\mathfrak{A}_0 : \mathfrak{U}_{\tilde{x}})}$$

for any $\mathfrak{A} \in \mathcal{B}(I)$.

Lemma II.11.2: The set function $\Phi_{\mathfrak{U}_{\tilde{x}}}$ is admissible and weakly additive on those sets separated by $\mathfrak{U}_{\tilde{x}}$ under [V].

Proof: (1) $\Phi_{\mathfrak{U}_{\tilde{x}}}(\mathfrak{A}) \geq 0$ is immediate for any $\mathfrak{A} \in \mathcal{B}(I)$.

(2) If $\mathfrak{A} \subseteq B$, then if $B \subseteq \bigcup_{j=1}^{n} V_j \mathfrak{U}_{\tilde{x}}$, $\mathfrak{A} \subseteq \bigcup_{j=1}^{n} V_j \mathfrak{U}_{\tilde{x}}$. Therefore $(\mathfrak{A} : \mathfrak{U}_{\tilde{x}}) \leq (B : \mathfrak{U}_{\tilde{x}})$ implying therefore that $\Phi_{\mathfrak{U}_{\tilde{x}}}(\mathfrak{A}) \leq \Phi_{\mathfrak{U}_{\tilde{x}}}(B)$.

(3) If $\mathfrak{A} \subseteq \bigcup_{n=1}^{n} V_j \mathfrak{U}_{\tilde{x}}$ and $B \subseteq \bigcup_{j=1}^{n'} V_j' \mathfrak{U}_{\tilde{x}}$, then $(\mathfrak{A} \cup B) \subseteq \left(\bigcup_{j=1}^{n} V_j \mathfrak{U}_{\tilde{x}}\right) \cup \left(\bigcup_{j=1}^{n'} V_j' \mathfrak{U}_{\tilde{x}}\right)$ and therefore

$\Phi_{\mathfrak{U}_{\tilde{x}}}(\mathfrak{A} \cup B) \leq \Phi_{\mathfrak{U}_{x}}(\mathfrak{A}) + \Phi_{\mathfrak{U}_{\tilde{x}}}(B)$ follows from the fact that $(\mathfrak{A} \cup B : \mathfrak{U}_{\tilde{x}}) \leq (\mathfrak{A} : \mathfrak{U}_{\tilde{x}}) + (B : \mathfrak{U}_{\tilde{x}})$.

For the second assertion, let $\mathfrak{U}_{\tilde{x}}$ separate $\mathfrak{A}$ from B under [V] and assume that $(\mathfrak{A} \cup B) \subseteq \bigcup_{j=1}^{n} V_j \mathfrak{U}_{\tilde{x}}$. By Definition II.8, for any V_j either $V_j \mathfrak{U}_{\tilde{x}} \cap \mathfrak{A} = \phi$ or $V_j \mathfrak{U}_{\tilde{x}} \cap B = \phi$. Then for some K, $\mathfrak{A} \subseteq \bigcup_{j=1}^{K} V_j \mathfrak{U}_{\tilde{x}}$ and $B \subseteq \bigcup_{j=K+1}^{n} V_j \mathfrak{U}_{\tilde{x}}$ and therefore $(\mathfrak{A} \cup B : \mathfrak{U}_{\tilde{x}}) = (\mathfrak{A} : \mathfrak{U}_{\tilde{x}}) + (B : \mathfrak{U}_{\tilde{x}})$ from whence $\Phi_{\mathfrak{U}_{\tilde{x}}}(\mathfrak{A} \cup B) = \Phi_{\mathfrak{U}_{\tilde{x}}}(\mathfrak{A}) + \Phi_{\mathfrak{U}_{\tilde{x}}}(B)$.

Q.E.D.

Lemma II.11.3: There exists an admissible, weakly additive extension of $\Phi_{\mathfrak{U}_{\tilde{x}}}$, ψ, that is defined on $\mathcal{B}(I)$ with range in R_{+}^{*}.

Proof: From Lemma II.11.2 and II.3, from the fact that $\Phi : \mathcal{B}(I) \times \{O_{\tilde{x}}\} \to R_+$ we have that $\Phi^* : \mathcal{B}^*(I^*) \times \{O^*_{\tilde{x}^*}\} \to R^*_+$. By Lemma II.5, there is an internal set $G \in \{O^*_{\tilde{x}^*}\}$ such that $G \subseteq \hat{u}(\tilde{x})$. Let $\psi(\mathfrak{A}) = \Phi^*_G(\mathfrak{A})$ for $\mathfrak{A} \in \mathcal{B}(I)$. Using Lemmas II.11.2 and II.3 once more, $\psi = \Phi^*_G$ is admissible since G is internal in $\{O^*_{\tilde{x}^*}\}$.

Let $\mathfrak{A}$ and B be separated in I under [V], then $\models (\forall D \in \{O_{\tilde{x}}\}) = \left[D \subseteq \mathfrak{U}_{\tilde{x}} \Rightarrow \Phi_D(\mathfrak{A} \cup B) = \Phi_D(\mathfrak{A}) + \Phi_D(B)\right]$. Then by Lemma II.3

$$* \models (\forall D^* \in \{O^*_{\tilde{x}^*}\}) \left(D^* \subseteq \mathfrak{U}^*_{\tilde{x}^*} \Rightarrow \Phi^*_{D^*}(\mathfrak{A} \cup B) = \Phi^*_{D^*}(\mathfrak{A}) + \Phi^*_{D^*}(B)\right)$$

However, one knows that $G \subseteq \hat{u}(\tilde{x}) \subseteq \mathfrak{U}^*_{\tilde{x}^*}$, and since G is internal, $G \in \{O^*_{\tilde{x}^*}\}$. Therefore,

$$\Phi^*_G(\mathfrak{A} \cup B) = \Phi^*_G(\mathfrak{A}) + \Phi^*_G(B)$$

which of course is simply

$$\psi(\mathfrak{A} \cup B) = \psi(\mathfrak{A}) + \psi(B).$$

Q.E.D.

Lemma II.11.4: For all $\mathfrak{A} \in \mathcal{B}(I)$, $\psi(A) \in M_0^+$, where $M_0^+ = \{r \in R^*_+ : r < n \text{ for } n \text{ standard and real}\}$.

Proof: Any member of $\mathcal{B}(I)$ is such that $A \subseteq F$ for some $F \in F_\sigma$. By condition (b) of the premises $\mathfrak{A} \subseteq F \subseteq \bigcup_{V \in [V]} V\mathfrak{A}_0$. Since F is compact on I, $\mathfrak{A} \subseteq F \subseteq \bigcup_{j=1}^{n} V_j\mathfrak{A}_0$ for some standard natural number $n \in N$ and $\{V_1, \ldots, V_n\} \subseteq [V]$. From the admissibility of ψ, $\psi(\mathfrak{A}) \leq \sum_{j=1}^{n} \psi(V_j\mathfrak{A}_0)$. And by construction of $\Phi_{\mathfrak{U}_{\tilde{x}}}$, since

$\Phi_{\mathfrak{U}_{\tilde{x}}}(\mathfrak{A}_0) = 1$, $\psi(\mathfrak{A}_0) = 1$. Then

$$\psi(A) \leq \sum_{j=1}^{n} \psi(V_j\mathfrak{A}_0) \leq \sum_{j=1}^{n} \psi(\mathfrak{A}_0) = n.$$

Q.E.D.

Remark: Implicit in the proof of Lemma II.11.4 is the fact that for any $V \in [V]$, $(V\mathfrak{A} : \mathfrak{U}_{\tilde{x}}) \leq (A : \mathfrak{U}_{\tilde{x}})$ which follows from the fact that if $\{V_1\mathfrak{U}_{\tilde{x}}, \ldots, V_n\mathfrak{U}_{\tilde{x}}\}$ is a covering of $\mathfrak{A} \in \mathfrak{B}(I)$, then $\{VV_1\mathfrak{U}_{\tilde{x}}, \ldots, VV_n\mathfrak{U}_{\tilde{x}}\}$ is a covering of $V\mathfrak{A}$.

Denote by st$(\cdot)$ the order preserving homomorphism from M_0^+ with Kernel M_1, taking unique values in R_+ as defined by Robinson [9], p. 57, termed the standard part. Then from the order preserving properties of st$(\cdot)$ along with Lemma II.11.3 note that st(ψ) is admissible and weakly additive on $\mathfrak{B}(I)$ with values in R_+. That $\text{st}(\psi(A)) \geq \text{st}(\psi(VA))$ for $V \in [V]$ and $\mathfrak{A} \in \mathfrak{B}(I)$ follows from the remark made following Lemma II.11.4. Since st(ψ) is real valued and nonnegative, defined on $\mathfrak{B}(I)$, allow $\lambda = \text{st}(\psi)$ to obtain the desideratum of the theorem.

Q.E.D.

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13. ABSTRACT

The problem of the existence of a subinvariant measure on a class of semi-group operators on the unit interval arising in Mathmatical Economics in the area of utility orderings over time [1,2] is given a nonstandard characterization. The result makes use of constructions found in the treatment of Hausner [5] and Parikh [6] of the construction of Haar measures of the real line.

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